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# ON GLUING OF QUASI-PSEUDOMETRIC SPACES

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Abstract. The concept of gluing a family of $T_0$-quasi-metric spaces along subsets was introduced by Otafudu. In this article, we continue the study of externally Isbell-convex and weakly externally Isbell-convex subsets of a $T_0$-quasi-metric space. We finally investigate some properties of the resulting $T_0$-quasi-metric space obtained by gluing a family of Isbell-convex $T_0$-quasi-metric spaces attached along isometric subspaces.

## 1. Introduction

How to glue a given family of metric spaces so that the resulting space satisfies properties of the given metric spaces is a classical problem that arises in geometry. The idea of constructing a new metric space from a family of metric spaces by gluing them together is well known in metric geometry. For a good overview about gluing a family of metric spaces, we refer the reader to [3].

Piatek [12] proved that if $(X, d_X)$ and $(Y, d_Y)$ are hyperconvex metric spaces such that $X \cap Y = [a, b]$, where $a$ and $b$ are connected by a unique metric segment in both $X$ and $Y$, then the metric space $(X \cup Y, d)$, where $d$ is defined by

$$d(x, y) = \min_{c\in[a,b]} [d_X(x, c) + d_Y(c, y)], \quad x \in X \setminus [a, b] \text{ and } y \in Y \setminus [a, b]$$

is hyperconvex too.

Recently Miesch [8] studied how to glue a family of hyperconvex metric spaces such that the resulting space remains hyperconvex. Miesch used ideas from [3] on gluing along metric spaces by attaching them along isometric subspaces. For

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instance, he proved that the resulting metric space $(X, m)$ from gluing a family $(X_i, m_i)_{i\in I}$ of hyperconvex metric spaces along a gated subset $A$ in $(X_i, m_i)$ whenever $i \in I$ is hyperconvex. More details about the theory of gated subsets of a metric space can be found in [4].

Hyperconvex metric spaces have been studied since 1956 (see [1]). A number of results on hyperconvex metric spaces have found applications in many fields of mathematics like geometry, topology, and operator theory. The importance of hyperconvex metric is not only limited to mathematics, they are also useful in other areas. For instance, the study of phylogenetic trees in biology and medicine also employs hyperconvex metric spaces (see for instance [4]). These are great motivations for extending results about hyperconvex metric spaces to the quasi-metric point of view.

Kemajou et al. [5] introduced successfully the concept of hyperconvexity in the setting of quasi-pseudometric spaces, which they called Isbell-convexity (or $q$-hyperconvexity). Olela Otafudu [11] introduced the concept of gated sets in a $T_0$-quasi-metric space called in-gated set. It turns out that in a $T_0$-quasi-metric space, there is a dual concept called outgated set. These concepts were then used to extend some classical results on gluing a family of hyperconvex metric spaces along a set such that the resulting space preserves hyperconvexity from a metric point of view to the quasi-metric setting. In this article, we continue the study of Isbell-convex quasi-metric spaces, in particular, we focus on externally Isbell-convex and weakly externally subsets of a $T_0$-quasi-metric space. Moreover, we extend the theory of gluing a family of metric spaces along isometric subsets to the context of quasi-pseudometric spaces. We finally study some properties of the resulting space from the gluing of a family of Isbell-convex quasi-metric spaces along isometric subspaces which are externally Isbell-convex and weakly externally Isbell-convex.

## 2. Convexity

In this section, we first recall some basic concepts on quasi-pseudometric spaces. We then study some interesting properties of externally Isbell-convex and weakly externally Isbell-convex subsets of a $T_0$-quasi-metric space.

**Definition 2.1.** Let $X$ be a nonempty set and let $d : X \times X \to [0, \infty)$ be a map. Then $d$ is a *quasi-pseudometric* on $X$ if

(a) $d(x, x) = 0$ whenever $x \in X$, and
(b) $d(x, z) \le d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

If $d$ is a quasi-pseudometric on a set $X$, then the pair $(X, d)$ is called a *a quasi-pseudometric space*. Moreover, we say that $d$ is a $T_0$-quasi-metric provided that it satisfies the additional condition that for any $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$. The set $X$ together with a $T_0$-quasi-metric on $X$ is called a *quasi-metric space*.

*Remark* 2.2. Note that if $d$ is a quasi-metric on $X$, then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric on $X$, called the *conjugate quasi-pseudometric* of $d$. As usual, a quasi-pseudometric

$d$ on $X$ such that $d = d^{-1}$ is called a *pseudometric* on $X$. Furthermore, the map $d^s = \max\{d, d^{-1}\}$ is a pseudometric on $X$. If $d$ is a $T_0$-quasi-metric on $X$, then $d^s$ is a metric on $X$.

Let $(X, d)$ be a quasi-pseudometric space and for each $x \in X$ and $r \in [0, \infty)$, let $C_d(x, r) = \{y \in X : d(x, y) \leq r\}$ be the $\tau(d^{-1})$-closed ball of center $x$ and radius $r$. Furthermore, the *open ball* with center $x$ and radius $r$ is represented by $B_d(x, r) = \{y \in X : d(x, y) < r\}$. If $A$ is a subset of $X$ and $r \in [0, \infty)$, then the set $C_d(A, r)$ defined by

$$C_d(A, r) = \{y \in X : \ \mathrm{dist}(A, y) = \inf_{x \in A} d(x, y) \leq r\}$$

is called $\tau(d^{-1})$-*closed* $r$-*neighborhood* of $A$.

**Definition 2.3** ([5, Definition 2])**.** A quasi-pseudometric space $(X, d)$ is called *Isbell-convex* (or *q-hyperconvex*) provided that for any family $(x_i)_{i \in I}$ of points in $X$ and families $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ of nonnegative real numbers satisfying $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$, the following condition holds:

$$\bigcap_{i \in I} [C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)] \neq \emptyset.$$

We next recall the definition of an externally Isbell-convex subset of a quasi-pseudometric space, which was introduced in [6].

**Definition 2.4** ([6, Definition 6.1])**.** Let $(X, d)$ be a quasi-pseudometric space. A subspace $E$ of $(X, d)$ is said to be *externally Isbell-convex* (or *externally q-hyperconvex*) (relative to $X$) if given any family $(x_i)_{i \in I}$ of points in $X$ and families of nonnegative real numbers $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$, the following condition holds:

If $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$ and $\mathrm{dist}(x_i, E) \leq r_i$ and $\mathrm{dist}(E, x_i) \leq s_i$ whenever $i \in I$, then $\bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \cap E \neq \emptyset$.

In what follows, we denote by $\mathcal{E}_q(X)$ the set of all externally Isbell-convex subsets of a quasi-pseudometric $(X, d)$.

For a $T_0$-quasi-metric space $(X, d)$, a subset $A$ of $X$ is called *q-admissible* if and only if it can be written as the intersection of a family of sets of the form $C_d(x, r) \cap C_{d^{-1}}(x, s)$, with $x \in X$ and $r, s \geq 0$. We shall denote the collection of all $q$-admissible subsets of $(X, d)$ by $\mathcal{A}_q(X)$.

The proof of the following lemma is easy and we leave it to the reader.

**Lemma 2.5.** *If $(X, d)$ is an Isbell-convex $T_0$-quasi-metric space, then $\mathcal{A}_q(X) \subseteq \mathcal{E}_q(X)$.*

**Lemma 2.6** ([6, Lemma 6.4])**.** *Let $(X, d)$ be an Isbell-convex quasi-pseudometric space. Suppose that $E \subseteq X$ is externally Isbell-convex relative to $X$ and that $A$ is a q-admissible subset of $(X, d)$ such that $E \cap A \neq \emptyset$. Then $E \cap A$ is externally Isbell-convex relative to $X$.*

**Lemma 2.7.** *Let $(X, d)$ be an Isbell-convex quasi-pseudometric space. If $A \in \mathcal{E}_q(X)$, then $C_d(A, r) \cap C_{d^{-1}}(A, s) \in \mathcal{E}_q(X)$ with $r, s \geq 0$.*

*Proof.* Let $A$ be an externally Isbell-convex subset of $X$ and let $C := C_d(A,r) \cap C_{d^{-1}}(A,s)$. Suppose a given family $(x_i)_{i\in I}$ of points in $X$ and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ satisfy $d(x_i,x_j) \leq r_i + s_j$ whenever $i,j \in I$ and $\text{dist}(x_i, C) \leq r_i$ and $\text{dist}(C, x_i) \leq s_i$ whenever $i \in I$.

By the external Isbell-convexity of $A$, we have

$$\bigcap_{i\in I}[C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i)] \cap A \neq \emptyset.$$

Let $a \in \bigcap_{i\in I}[C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i)] \cap A$. Then

$$\begin{aligned} d(x_i,a) &\leq r_i + s \text{ and} \\ d(a,x_i) &\leq s_i + r \text{ for all } i \in I. \end{aligned}$$

This implies that

$$\begin{aligned} \text{dist}(x_i,A) &\leq r_i + s \text{ and} \\ \text{dist}(A,x_i) &\leq s_i + r \text{ for all } i \in I. \end{aligned}$$

By the external Isbell-convexity of $A$ we have

$$\bigcap_{i\in I}[C_d(x_i,r_i+s) \cap C_{d^{-1}}(x_i,s_i+r)] \cap C \neq \emptyset.$$

Let $y \in \bigcap_{i\in I}[C_d(x_i,r_i+s) \cap C_{d^{-1}}(x_i,s_i+r)] \cap C$; then

$$d(x_i,y) \leq r_i + s$$

and

$$d(y,x_i) \leq s_i + r.$$

Hence

$$\begin{aligned} \emptyset \neq \bigcap\nolimits_{i\in I}&[C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i)) \cap (C_d(y,r) \cap C_{d^{-1}}(y,r)] \\ &\subseteq \bigcap\nolimits_{i\in I}[C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i)] \cap C. \end{aligned}$$

□

The following theorem is the asymmetric version of the well-known Baillon's theorem [2], which is useful in the rest of the paper.

**Theorem 2.8** ([6, Theorem 4.1])**.** *Let $(X,d)$ be a bounded Isbell-convex $T_0$-quasi-metric space. Moreover, let $(X_i)_{i\in I}$ be a descending family of nonempty externally Isbell-convex subsets of $X$, where $I$ is assumed to be totally ordered such that $i_1,i_2 \in I$ and $i_1 \leq i_2$ if and only if $X_{i_2} \subseteq X_{i_1}$. Then $\bigcap_{i\in I} X_i$ is nonempty and externally Isbell-convex relative to $X$.*

**Proposition 2.9.** *If $(X_i)_{i\in I}$ is a family of bounded externally Isbell-convex subsets of a $T_0$-quasi-metric space $(X,d)$ such that $\bigcap_{i\in J} X_i$ is nonempty and externally Isbell-convex whenever $J \subseteq I$ is finite, then the intersection $\bigcap_{i\in I} X_i$ is nonempty and externally Isbell-convex.*

*Proof.* Consider $\Lambda = \{K \subseteq I : \text{for all } J \text{ finite, } J \subseteq I, \bigcap_{i\in K\cup J} X_i \text{ is nonempty and externally Isbell-convex}\}$. Obviously, $\emptyset \in \Lambda$ and $\Lambda$ satisfies the hypothesis of Zorn's lemma because of Theorem 2.8. Let $K$ be maximal in $\Lambda$. Then, $K \cup \{i\} \in \Lambda$ whenever $i \in I$. Because of the maximality of $J$, we have $i \in K$ whenever $i \in I$. □

The following proposition will be useful in what follows.

**Proposition 2.10** ([6, Proposition 4.9])**.** *Let $(X, d)$ be a $T_0$-quasi-metric space. If $Y$ is an externally Isbell-convex subset of $(X, d)$ and $A$ is an externally Isbell-convex relative to $Y$, then $A$ is also externally Isbell-convex relative to $X$.*

*Proof.* Let a family $(x_i)_{i\in I}$ of points in $X$ and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ satisfy $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$ and $\text{dist}(x_i, A) \leq r_i$ and $\text{dist}(A, x_i) \leq s_i$ whenever $i \in I$. Then for $i \in I$, the set $A_i = (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \cap Y$ is an externally Isbell-convex subset of $X$ and $Y$ by Lemma 2.6.

It is easy to see that $A_i \cap A \neq \emptyset$ whenever $i \in I$ and

$$A_i \cap A_j = (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \cap (C_d(x_j, r_j) \cap C_{d^{-1}}(x_j, s_j)) \cap Y \neq \emptyset$$

by the external Isbell-convexity of $Y$.

Hence $A \cap \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) = A \cap \bigcap_{i\in I} A_i \neq \emptyset$ by Lemma 2.9. □

**Definition 2.11.** Let $(X, d)$ be a quasi-pseudometric space. A subspace $E$ of $(X, d)$ is said to be *weakly externally Isbell-convex* (relative to $X$) if $E$ is externally Isbell-convex relative to $E \cup \{z\}$ for each $z \in X$. Precisely, given any family $(x_i)_{i\in I}$ of points in $X$ all but at most one of which lies in $E$, and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ satisfying $d(x_i, x_j) \leq r_i + s_j$, with $\text{dist}(x_i, E) \leq r_i$ and $\text{dist}(E, x_i) \leq s_i$ if $x_i \notin E$, it follows that $\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \cap E \neq \emptyset$.

**Example 2.12.** Let $X = [0, \infty)$ be the set of nonnegative reals equipped with the $T_0$-quasi-metric $u(x, y) = \max\{x - y, 0\}$. Then $D = [0, 1] \subseteq X$ is weakly externally Isbell-convex relative to $X$.

**Example 2.13** ([10, Example 4.3])**.** Consider the Isbell-convex $T_0$-quasi-metric space $(\mathbb{R}, u)$, where $u(x, y) = \max\{x - y, 0\}$ whenever $x, y \in \mathbb{R}$. Then $A = C_u(2, 0) \cap C_{u^{-1}}(2, 0)$ is not weakly externally Isbell-convex relative to $[0, \infty)$ but it is externally Isbell-convex relative to $\mathbb{R}$.

If $(X, d)$ is $T_0$-quasi-metric space, then in what follows, we will denote by $\mathcal{W}_q(X)$ the collection of all weakly externally Isbell-convex subsets of $(X, d)$.

**Proposition 2.14.** *Let $(X, d)$ be an Isbell-convex $T_0$-quasimetric space and $A \subseteq X$. Let $(x_i)_{i\in I}$ be a family of points in $X$ and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be two families of nonnegative real numbers. If $A = \bigcap_{i\in I} C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)$ and $r, s \geq 0$, then $C_d(A, s) \cap C_{d^{-1}}(A, r) = \bigcap_{i\in I} C_d(x_i, r_i + s) \cap C_{d^{-1}}(x_i, s_i + r) \in \mathcal{A}_q(X)$.*

*Proof.* Let $y \in C_d(A, s) \cap C_{d^{-1}}(A, r)$; then $\text{dist}(A, y) \leq s$ and $\text{dist}(y, A) \leq r$. Moreover, we have $d(t, y) \leq s$ and $d(y, t) \leq r$ for some $t \in A$. Then

$$d(x_i, y) \leq d(x_i, t) + d(t, y) \leq r_i + s,$$

and

$$d(y, x_i) \leq d(y, t) + d(t, x_i) \leq r + s_i,$$

whenever $i \in I$. Hence $y \in \bigcap_{i \in I} C_d(x_i, r_i + s) \cap C_{d^{-1}}(x_i, s_i + r)$.

Conversely, let $z \in \bigcap_{i \in I} C_d(x_i, r_i + s) \cap C_{d^{-1}}(x_i, s_i + r)$. Then

$$d(x_i, z) \leq r_i + s$$

and

$$d(z, x_i) \leq s_i + r$$

whenever $i \in I$. By the Isbell-convexity of $(X, d)$, we have

$$\emptyset \neq C_d(z, r) \cap C_{d^{-1}}(z, s) \cap \bigcap_{i \in I} C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) = C_d(z, r) \cap C_{d^{-1}}(z, s) \cap A.$$

Let $t \in C_d(z, r) \cap C_{d^{-1}}(z, s) \cap A$; then $d(z, t) \leq r$ and $d(t, z) \leq s$. Therefore,

$$z \in C_d(A, s) \cap C_{d^{-1}}(A, r).$$

□

**Proposition 2.15** (Compare [9, Lemma 2.3])**.** *Let $(X, d)$ be an Isbell-convex $T_0$-quasimetric space and let $A \subseteq X$. Then $A$ is weakly externally Isbell-convex if and only if for any $x \in X$, and by setting* $\operatorname{dist}(A, x) = s$ *and* $\operatorname{dist}(x, A) = r$, *the following conditions hold:*

(i) *The set $C_d(x, s) \cap C_{d^{-1}}(x, r) \cap A$ is externally Isbell-convex in $A$.*

(ii) *For any $y \in A$, there exist $a, a' \in C_d(x, s) \cap C_{d^{-1}}(x, r) \cap A$ such that*

$$d(x, y) = d(x, a) + d(a, y)$$

*and*

$$d(y, x) = d(y, a') + d(a', x).$$

*Proof.* Suppose that $A$ is weakly externally Isbell-convex. Then $C_d(x, s) \cap C_{d^{-1}}(x, r) \cap A$ is externally Isbell-convex in $A$ as an intersection of a $q$-admissible and an externally Isbell-convex subset of an Isbell-convex $T_0$-quasi-metric space $(X, d)$. Furthermore, for some $y \in A$ and since $\operatorname{dist}(A, x) = s$ and $\operatorname{dist}(x, A) = r$, we have $d(y, x) \geq s$ and $d(x, y) \geq r$. It follows that

$$d(x, y) = d(x, y) - r + r$$

and

$$d(y, x) = s + d(y, x) - s.$$

By the weak external Isbell-convexity of $A$, there exists

$$\emptyset \neq C_d(x, r) \cap C_{d^{-1}}(y, d(x, y) - r) \cap A$$

and there exists

$$\emptyset \neq C_{d^{-1}}(x, s) \cap C_d(y, d(y, x) - s) \cap A.$$

We observe that the family

$$\{C_d(y, d(y, x) - s)_{y \in A}, C_{d^{-1}}(y, d(x, y) - r)_{y \in A}, C_{d^{-1}}(x, s), C_d(x, r)\}$$

is Isbell-complete. Then by the Isbell-convexity of $(X, d)$, there exists

$$a \in C_d(y, d(y, x) - s) \cap C_{d^{-1}}(y, d(x, y) - r) \cap C_d(x, r) \cap C_{d^{-1}}(x, s) \cap A.$$

Then

$$d(x,y) \le d(x,a) + d(a,y) \le d(x,y) - r + r = d(x,y)$$

and

$$d(y,x) \le d(y,a) + d(a,x) \le s + d(y,x) - s = d(y,x).$$

So $d(x,y) = d(x,a) + d(a,y)$. By similar arguments, one shows that for

$$a' \in C_d(y, d(y,x) - s) \cap C_{d^{-1}}(y, d(x,y) - r) \cap C_d(x,r) \cap C_{d^{-1}}(x,s) \cap A.$$

such that $d(y,x) = d(y,a') + d(a',x)$.

Conversely, assume that (i) and (ii) are satisfied; then we prove that $A$ is weakly externally Isbell-convex. Let $(x_i)_{i\in I}$ be a family of points in $X$ and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be two families of nonnegative real numbers, satisfying $d(x_i,x_j) \le r_i + s_j$, with $\text{dist}(x_i, A) \le r_i$ and $\text{dist}(A, x_i) \le s_i$. Then by the Isbell-convexity of $X$, we have $\bigcap_{i\in I}(C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i)) \neq \emptyset$. If $x \in \bigcap_{i\in I}(C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i))$, then for $x_i \in A$, we have

$$\text{dist}(x, C_d(x, \text{dist}(x,A)) \cap C_{d^{-1}}(x, \text{dist}(A,x)) \cap A) \le r_i$$

and

$$\text{dist}(C_d(x, \text{dist}(x,A)) \cap C_{d^{-1}}(x, \text{dist}(A,x)) \cap A, x) \le s_i.$$

It follows that

$$\emptyset \neq \Big(A \cap C_d(x, \text{dist}(x,A)) \cap C_{d^{-1}}(x, \text{dist}(A,x))\Big) \cap \Big(\bigcap_{i\in I}(C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i))\Big).$$

If $x \in X$, $\text{dist}(A,x) = s \le r_1$ and $\text{dist}(x,A) = r \le r_2$, then $C_d(x,s) \cap C_{d^{-1}}(x,r) \cap A = C_d(C_d(x,s) \cap C_{d^{-1}}(x,r) \cap A, s - r_1) \cap C_{d^{-1}}(C_d(x,s) \cap C_{d^{-1}}(x,r) \cap A, r - r_1)$. Therefore, $C_d(x,s) \cap C_{d^{-1}}(x,r) \cap A$ is externally Isbell-convex by Proposition 2.14 and Lemma 12. Furthermore, by (ii) for $x_i \in A$, $d(x,x_i) \le s + r_i$ and $d(x_i,x) \le r_i + s_i$, we have $\text{dist}(x_i, C_d(x,s) \cap C_{d^{-1}}(x,r) \cap A) \le s_i$ and $\text{dist}(C_d(x,s) \cap C_{d^{-1}}(x,r) \cap A, x_i) \le r_i$. Therefore,

$$A \cap \Big(C_d(x,s) \cap C_{d^{-1}}(x,r)\Big) \cap \Big(\bigcap_{i\in I}(C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i))\Big) \neq \emptyset.$$

□

**Lemma 2.16** (Compare [9, Lemma 2.4])**.** *Let $(X,d)$ be a $T_0$-quasi-metric space, let $A \in \mathcal{W}_q(X)$, and let $r,s \ge 0$. Then there exists a retraction $\varphi : C_d(A,s) \cap C_{d^{-1}}(A,r) \longrightarrow A$ such that $d(x,\varphi(x)) \le s$ and $d(\varphi(x),x) \le r$ whenever $x \in C_d(A,s) \cap C_{d^{-1}}(A,r)$.*

*Proof.* Consider

$$\mathcal{F} := \{(C,\varphi) : A \subseteq C \subseteq C_d(A,s) \cap C_{d^{-1}}(A,r) \text{ and } \varphi : C \to A \text{ is a retraction}$$
$$\text{such that } d(x,\varphi(x)) \le s \text{ and } d(\varphi(x),x) \le r \text{ whenever } x \in C\}.$$

Let Id denote the identity map on $A$. Then $(A, \text{Id}) \in \mathcal{F}$. So $\mathcal{F} \neq \emptyset$. If we partially order $\mathcal{F}$ by $((C_1,\varphi_1) \preceq (C_2,\varphi_2)$ if and only if $C_1 \subseteq C_2$ and $\varphi_2$ is an extension of $\varphi_1)$, then each chain of $(\mathcal{F}, \preceq)$ is bounded above. So by Zorn's lemma, $\mathcal{F}$ has a maximal element. Let the maximal element of $\mathcal{F}$ be $(\overline{C}, \overline{\varphi})$. We have to show that $\overline{C} = C_d(A,s) \cap C_{d^{-1}}(A,r)$. Suppose that there exists $x \in \overline{C}$ such that $x \notin \overline{C}$. Then for any $y \in \overline{C}$ we set $r_y = d(x,y)$ and $s_y = d(y,x)$. Then

$$d(x,\overline{\varphi}(y)) \leq d(x,y) + d(y,\overline{\varphi}(y)) \leq r_y + s$$

and

$$d(\overline{\varphi}(y),x) \leq d(\overline{\varphi}(y),y) + d(y,x) \leq r + s_y.$$

By the weak external Isbell-convexity of $A$, we have

$$z \in C_d(x,s) \cap C_{d^{-1}}(x,r) \cap \left[\bigcap_{y\in\overline{C}} C_d(\overline{\varphi}(y),s_y) \cap C_{d^{-1}}(\overline{\varphi}(y),r_y)\right] \cap A.$$

Let $y \in \overline{C}$. We define $\varphi^{'} : \overline{C} \cup \{z\} \longrightarrow A$ by $\varphi^{'}(y) = \overline{\varphi}(y)$ if $y \in \overline{C}$ and $\varphi^{'}(y) = z$. Then, for each $y \in \overline{C}$, we have

$$d(\varphi^{'}(y),y) = d(\overline{\varphi}(y),y) \leq r$$

and

$$d(y,\varphi^{'}(y)) = d(y,\overline{\varphi}(y)) \leq s.$$

Therefore, the pair $(\overline{C} \cup \{z\}, \varphi^{'})$ contradicts the maximality of $(\overline{C},\overline{\varphi})$ in $(\mathcal{F},\preceq)$. So $\overline{C} = C_d(A,s) \cap C_{d^{-1}}(A,r)$. □

**Lemma 2.17.** *Let $A$ be a weakly externally Isbell-convex subset of an Isbell-convex $T_0$-quasi-metric space $(X,d)$. Then for any $r,s \geq 0$ and for all $x \in X$, we have that $C_d(x,\mathrm{dist}(A,x)) \cap C_{d^{-1}}(x,\mathrm{dist}(x,A)) \cap A$ is an externally Isbell-convex subset of $X$. Moreover, for all $x \in X$, the set $C_d(x,\mathrm{dist}(x,C)) \cap C_{d^{-1}}(x,\mathrm{dist}(C,x)) \cap C$ is externally Isbell-convex too, where $C := C_d(A,s) \cap C_{d^{-1}}(A,r)$, $r = \mathrm{dist}(x,C)$ and $s = \mathrm{dist}(C,x)$.*

*Proof.* Let $x \in X$. Consider a family $(x_i)_{i\in I}$ of points in $C_d(A,s) \cap C_{d^{-1}}(A,r)$ and two families $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ of nonnegative real numbers such that $d(x_i,x_j) \leq r_i + s_j$, $d(x_i,x) \leq s + r_i$ and $d(x,x_i) \leq r + s_i$ with

$$\mathrm{dist}(C_d(A,s) \cap C_{d^{-1}}(A,r),x) = s$$

and

$$\mathrm{dist}(x,C_d(A,s) \cap C_{d^{-1}}(A,r)) = r.$$

By Lemma 2.16, there exists a retraction $\varphi : C_d(A,s) \cap C_{d^{-1}}(A,r) \to A$ such that $d(y,\varphi(y)) \leq s$ and $d(\varphi(y),y) \leq r$ whenever $y \in C_d(A,s) \cap C_{d^{-1}}(A,r)$. Then it follows that

$$d(\varphi(x_i),x) \leq d(\varphi(x_i),x_i) + d(x_i,x) \leq r + s + r_i$$

and

$$d(x,\varphi(x_i)) \leq d(x,x_i) + d(x_i,\varphi(x_i)) \leq r + s_i + s$$

whenever $i \in I$. Then there exists

$$y \in C_{d^s}(x,r+s) \cap \left[\bigcap_{i\in I} C_d(\varphi(x_i),r_i) \cap C_{d^{-1}}(\varphi(x_i),s_i)\right] \cap A.$$

Thus $d(x,y) \leq r+s$, $d(y,x) \leq r+s$ and

$$d(x_i,y) \leq d(x_i,\varphi(x_i)) + d(\varphi(x_i),y) \leq s + r_i$$

and

$$d(y, x_i) \leq d(y, \varphi(x_i)) + d(\varphi(x_i), x_i) \leq r + s_i$$

whenever $i \in I$. By the Isbell-convexity of $(X, d)$, we have

$$\emptyset \neq C_d(x,r) \cap C_{d^{-1}}(x,s) \cap \left[\bigcap_{i\in I} C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i)\right] \cap C_d(y,r) \cap C_{d^{-1}}(y,s)$$

$$\subseteq C_d(x,r) \cap C_{d^{-1}}(x,s) \cap \left[\bigcap_{i\in I} C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i)\right] \cap C_d(A,s) \cap C_{d^{-1}}(A,r).$$

Hence $C_d(A,s) \cap C_{d^{-1}}(A,r)$ is externally Isbell-convex in $X$.

Furthermore, we have

$$\begin{aligned} &C_d(x,r) \cap C_{d^{-1}}(x,s) \cap C_d(A,s) \cap C_{d^{-1}}(A,r) \\ &\quad = C_d(x,r) \cap C_{d^{-1}}(x,s) \cap \Big[ C_d[C_d(x, \operatorname{dist}(x,A)) \cap C_{d^{-1}}(x, \operatorname{dist}(A,x)) \cap A), s] \\ &\qquad \cap C_{d^{-1}}[C_d(x, \operatorname{dist}(x,A)) \cap C_{d^{-1}}(x, \operatorname{dist}(A,x) \cap A), r)] \Big]. \end{aligned}$$

If $C_d(x, \operatorname{dist}(x,A)) \cap C_{d^{-1}}(x, \operatorname{dist}(A,x) \cap A$ is externally Isbell-convex, then $C_d(x,r) \cap C_{d^{-1}}(x,s) \cap C_d(A,s) \cap C_{d^{-1}}(A,r)$ is also externally Isbell-convex. □

**Lemma 2.18.** *Let $(X,d)$ be a $T_0$-quasi-metric space. If $Y \in \mathcal{W}_q(X)$ and $A \in \mathcal{E}_q(X)$, then $A \in \mathcal{W}_q(X)$.*

*Proof.* Let a family $(x_i)_{i\in I}$ of points in $X$ and families $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ of non-negative real numbers satisfying $d(x_i,x_j) \leq r_i + s_j$ for all $i,j \in I$, $\operatorname{dist}(x_i, A) \leq r_i$ and $\operatorname{dist}(A, x_i) \leq s_i$ for all $i \in I$ be given. Then we have for every $i \in I$, the set

$$A_i = C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) \cap Y$$

is externally Isbell convex by Lemma 2.6. Furthermore, for every $i \in I$, $A_i$ is also externally Isbell convex relative to $Y$. Since for every $i \in I$, $A_i \cap A \neq \emptyset$ and $Y$ is externally Isbell convex, we have

$$A_i \cap A_j = C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i) \cap C_d(x_j,r_j) \cap C_{d^{-1}}(x_j,s_j) \cap Y \neq \emptyset. \quad (2.1)$$

Therefore, we have pairwise intersecting externally Isbell convex subsets of $Y$ and by Proposition 2.9,

$$A \cap \bigcap_{i\in I} C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i) = A \cap_{i\in I} A_i \neq \emptyset.$$

□

## 3. Gluing of quasi-pseudometric spaces

The theory of amalgamation of two finite $T_0$-quasi-metric spaces has been introduced in [7]. In this section, we extend the amalgamation of two finite $T_0$-quasi-metric space to any family of $T_0$-quasi-metric spaces.

**Proposition 3.1** ([11, Proposition 8]). *Let $(X_\alpha, d_\alpha)_{\alpha\in\Gamma}$ be a family of $T_0$-quasi-metric spaces and let $(A, d_A)$ be a $T_0$-quasi-metric space. If $A_\alpha \subseteq X_\alpha$ and fix some isometry $\varphi_\alpha : A \to A_\alpha$ whenever $\alpha \in \Gamma$, then there exists a $T_0$-quasi-metric space $X = \bigsqcup_A X_\alpha$, the coproduct of $X_\alpha$ amalgamated along $A$ or $\varphi_\alpha$ such that for all $a \in A$, $\varphi_\alpha(a)$ coincides with $a$ in $X$: Making use of this identification between the elements of $A_\alpha$ whenever $\alpha \in \Gamma$, for $x \in X_\alpha$ and $y \in X_{\alpha'}$ with $\alpha \neq \alpha'$, we set the $T_0$-quasi-metric $d$ on $X$ by*

$$d(x, y) = \inf_{a\in A}\{d_\alpha(x, \varphi_\alpha(a)) + d_{\alpha'}(\varphi_{\alpha'}(a), y)\}$$

*and*

$$d(y, x) = \inf_{a\in A}\{d_{\alpha'}(y, \varphi_{\alpha'}(a)) + d_\alpha(\varphi_\alpha(a), x)\},$$

*the subspaces $X_\alpha$ of $X$ carry their $T_0$-quasi-metrics $d_\alpha$, respectively.*

**Definition 3.2** (Compare [8, Definition 2.1]). Let $(X_\alpha, d_\alpha)_{\alpha\in\Gamma}$ be a family of $T_0$-quasi-metric spaces and let $(A, d_A)$ be a $T_0$-quasi-metric space. If $A_\alpha \subseteq X_\alpha$ and fix some isometry $\varphi_\alpha : A \to A_\alpha$ whenever $\alpha \in \Gamma$. If $(X, d)$ is the coproduct of $X_\alpha$ amalgamated along $A$ or $\varphi_\alpha$, then we call the $T_0$-quasi-metric space $(X, d)$ the *gluing* of $(X_\alpha, d_\alpha)_{\alpha\in\Gamma}$ along $A$ or $\varphi_\alpha$.

**Proposition 3.3.** *Let $(X_\alpha, d_\alpha)_{\alpha\in\Gamma}$ be a family of Isbell-convex $T_0$-quasi-metric spaces and let $A$ be an externally Isbell-convex subset relative to $(X_\alpha, d_\alpha)$ whenever $\alpha \in \Gamma$. If $(X, d)$ is the $T_0$-quasi-metric space obtained by gluing the family $(X_\alpha, d_\alpha)_{\alpha\in\Gamma}$ of Isbell-convex $T_0$-quasi-metric spaces along the set $A$, then for $\mathrm{dist}_d(x, A) = r$ and $\mathrm{dist}_d(A, x) = s$, there exists $a \in A \cap [C_{d_\alpha}(x, r) \cap C_{d_\alpha^{-1}}(x, s)]$ such that*

$$d(x, y) = d(x, a) + d(a, y)$$

*and*

$$d(y, x) = d(y, a) + d(a, x)$$

*whenever $x \in X_\alpha$ and $y \in X_{\alpha'}$ with $\alpha \neq \alpha'$.*

*Proof.* Consider the set $A' = A \cap [C_{d_\alpha}(x, r) \cap C_{d_\alpha^{-1}}(x, s)] \neq \emptyset$.

For any $\alpha \in \Gamma$, we have

$$d_\alpha(x, a) = d_\alpha(x, a) - r + r,$$

and

$$d_\alpha(a, x) = s + d_\alpha(a, x) - s,$$

then

$$A'' = A \cap [C_{d_\alpha}(x, r) \cap C_{d_\alpha^{-1}}(x, s)] \cap [C_{d_\alpha}(a, d_\alpha(a, x) - s) \cap C_{d_\alpha^{-1}}(a, d_\alpha(x, a) - r)] \neq \emptyset,$$

by the external Isbell-convexity of $A$.

Let $a' \in A''$. Then

$$d_\alpha(x, a') \leq d_\alpha(x, a) - r \text{ and } d_\alpha(a', a) \leq r, \tag{3.1}$$

and

$$d_\alpha(a, a') \leq s \text{ and } d_\alpha(a', x) \leq d_\alpha(a, x) - s. \tag{3.2}$$

Moreover, for $a \in A^{'} = A \cap [C_{d_\alpha}(x,r) \cap C_{d_\alpha^{-1}}(x,s)]$ and $a^{'} \in A^{''}$. From inequalities in (3.1), we have

$$d_\alpha(x,a^{'}) + d_{\alpha'}(a^{'},y) \le d_\alpha(x,a^{'}) + d_{\alpha'}(a^{'},a) + d_{\alpha'}(a,y) \le d_\alpha(x,a) + d_\alpha(a,y).$$

Therefore,

$$d(x,y) = \inf_{a\in A^{'}}\{d_\alpha(x,a) + d_{\alpha'}(a,y)\} = \text{dist}_\alpha(x,A^{'}) + \text{dist}_{\alpha'}(A^{'},y).$$

Hence

$$d(x,y) = r + \text{dist}_{\alpha'}(A^{'},y),$$

since $\text{dist}_\alpha(x,A^{'}) = \text{dist}_\alpha(x,A) = r$.

By similar arguments and inequalities in (3.2), we have

$$d(y,x) = \text{dist}_{\alpha'}(y,A^{'}) + s.$$

It follows that $A^{'}$ is externally Isbell-convex by Lemma 2.6 and $A$ is an externally Isbell-convex set relative to $X_\alpha$ whenever $\alpha \in \Gamma$. Moreover, since $A^{'} \subseteq A$ then by Proposition 2.10, we have that $A^{'}$ is externally Isbell-convex relative to $X_\alpha$ whenever $\alpha \in \Gamma$. Thus

$$\emptyset \ne C = [C_d(x,r) \cap C_{d^{-1}}(x,s)] \cap [C_d(y, \text{dist}_{\alpha'}(A^{'},y)) \cap C_{d^{-1}}(y, \text{dist}_{\alpha'}(y,A^{'}))].$$

If $a \in C$, then

$$d(x,y) \le d(x,a) + d(a,y) \le \text{dist}_\alpha(x,A^{'}) + \text{dist}_{\alpha'}(A^{'},y) \ge d(x,y).$$

Hence, $d(x,y) = d(x,a) + d(a,y)$.

Similarly, we have $d(y,x) = d(y,a) + d(a,x)$. □

**Proposition 3.4.** *Let $(X_\alpha, d_\alpha)_{\alpha\in\Gamma}$ be a family of Isbell-convex $T_0$-quasi-metric spaces and let $A$ be a weakly externally Isbell-convex subset relative to $(X_\alpha, d_\alpha)$ whenever $\alpha \in \Gamma$. If $(X,d)$ is the $T_0$-quasi-metric space obtained by gluing the family $(X_\alpha, d_\alpha)_{\alpha\in\Gamma}$ of Isbell-convex $T_0$-quasi-metric spaces along the set $A$, then there exist points $a \in C_d(x, \text{dist}(x,A)) \cap C_{d^{-1}}(x, \text{dist}(A,x)) \cap A$ and $a' \in C_d(x', \text{dist}(x',A)) \cap C_{d^{-1}}(x', \text{dist}(A,x')) \cap A$ such that*

$$d(x,x') = d(x,a) + d(a,a') + d(a',x'),$$

*whenever $x \in X_\alpha$ and $x' \in X_{\alpha'}$.*

*Proof.* Suppose that $A$ is weakly externally Isbell-convex in $X_\alpha$ whenever $\alpha \in \Gamma$. Then there exist

$$a \in C_d(x, \text{dist}(x,A)) \cap C_{d^{-1}}(x, \text{dist}(A,x)) \cap A$$

and

$$a' \in C_d(x', \text{dist}(x',A)) \cap C_{d^{-1}}(x', \text{dist}(A,x')) \cap A$$

such that $d(x,y) = d(x,a) + d(a,y)$ and $(y,x') = d(y,a') + d(a',x')$ by Proposition 2.15. Let $C := C_d(x, \text{dist}(x,A)) \cap C_{d^{-1}}(x, \text{dist}(A,x)) \cap A$ and let $C' := C_d(x', \text{dist}(x',A)) \cap C_{d^{-1}}(x', \text{dist}(A,x')) \cap A$.

Then

$$d(x,x') \le d(x,a) + d(a,a') + d(a,x') \le \text{dist}(x,A) + \text{dist}(C,C') + \text{dist}(A,x'). \quad (3.3)$$

Furthermore,

$$\mathrm{dist}(x, A) + \mathrm{dist}(C, C') + \mathrm{dist}(A, x') \geq d(x, x') \tag{3.4}$$

by the triangle inequality and taking the infimum on $C$ and $C'$. Combining (3.3) and (3.4), we have

$$d(x, x') = d(x, a) + d(a, a') + d(a', x').$$

□

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## References

1. N. Aronszajn and P. Panitchpakdi, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956) 405–439.
2. J.B. Baillon, *Nonexpansive mappings and hyperconvex spaces*, Contemp. Math. **72** (1988) 11–19.
3. M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren Math. Wiss. 319, Springer-Verlag, Berlin, 1999.
4. A.W.M. Dress and R. Scharlau, *Gated sets in metric spaces*, Aequationes Math. **34** (1987), no. 1, 112–120.
5. E. Kemajou, H.-P.A. Künzi and O. Olela Otafudu, *The Isbell-hull of a di-space,* Topology Appl. **159** (2012), no. 9, 2463–2475.
6. H.-P. Künzi and O. Olela Otafudu, *q-hyperconvexity in quasi-pseudometric spaces and fixed point theorems,* J. Funct. Spaces Appl. **2012** (2012), Art. ID 765903, 18 pp.
7. H.-P. Künzi and M. Sanchis, *The Kâtatov construction modified for a $T_0$-quasi-metric space*, Topology Appl. **159** (2012) 711–720.
8. B. Miesch, *Gluing hyperconvex metric spaces,* Anal. Geom. Metr. Spaces **3** (2015), no. 1, 102–110.
9. B. Miesch and M. Pavón, *Weakly externally hyperconvex subsets and hyperconvex gluings*, J. Topol. Anal. **9** (2017), no. 3, 379–407.
10. O. Olela Otafudu, *On weak external q-hyperconvexity,* Ann. Funct. Anal. **6** (2015), no. 3, 96–109.
11. O. Olela Otafudu, *Gated sets in quasi-metric spaces,* Topology Appl. **263** (2019) 159–171.
12. B. Piatek, *On the gluing of hyperconvex metrics and diversities,* Ann. Univ. Paedagog. Crac. Stud. Math. **13** (2014) 65–76.

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